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Attainable conditions and exact invariant for the time-dependent harmonic oscillator

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Abstract

The time-dependent oscillator equation is solved numerically for various trajectories in amplitude and phase variables. The solutions exhibit a finite time-dependent parameter whenever the squared amplitude times the derivative of the phase is invariant. If the invariant relationship does not hold, the time-dependent parameter has divergent singularities. These observations lead to the proposition that the harmonic oscillator equation with finite time-dependent parameter must have amplitude and phase solutions fulfilling the invariant relationship. Since the time-dependent parameter or the potential must be finite for any real oscillator implementation, the invariant must hold for any such physically realizable system.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The second-order linear non-autonomous differential equation

$$\frac{\mathrm{d}^2 q(t)}{\mathrm{d}t^2} + \Omega^2(t) q(t) = 0 \tag{1}$$

has received considerable attention in classical time-dependent oscillators, quantum mechanics, one-dimensional wave propagation and many other physical systems [1]. If we consider the equation to represent the time evolution of an oscillator with coordinate $q(t)$, the energy of the oscillator is not a conserved quantity when the parameter Ω^2 is time dependent. In order to describe the evolution of the oscillator for a slowly varying parameter, the theory of adiabatic invariants was developed at the beginning of the 20th century. In a rather more mathematical context, exact invariants for these types of equations were developed by Ermakov [2] since the late 19th century. However, his work did not permeate into the physics community until the last decades of the past century due to the work initiated by Lewis [3]. Since then, there has been increasing interest on the theory of exact invariants because they are constants of motion even if the time-dependent parameter varies rapidly compared with the characteristic period of the system.

On the other hand, the amplitude and phase representation of the coordinate variable has become widespread in oscillatory classical mechanics problems [4] and almost concomitant to the representation of the disturbance in wave phenomena [5]. Although this representation leads to nonlinear equations for the amplitude and the phase it has proved quite useful for tackling time-dependent mechanical problems. These two ingredients, namely an invariant and the amplitude/phase representation, have led to efficient numerical procedures often referred to as the phase integral method and Milne's amplitude approach [6]. This scheme has also permitted novel analytical solutions for sub-wavelength monotonic time-dependent parameters [7]. The comparison between the analytical solutions with their numerical and experimental counterparts has led to the present paper.

The central issue addressed in this paper is the validity of the orthogonal functions' exact invariant and its relationship with the behaviour of the time-dependent parameter. To this end, we evaluate in the next section the time-dependent parameter arising from different trajectories in the amplitude and phase representation without fulfilling the invariant relationship. It is noted that the parameter always involves singularities even if the trajectories are smooth. In section 3, the parameter is then re-evaluated but imposing the invariant relationship on the amplitude and phase variables. The parameter is then smooth and no longer exhibits awkward divergences. These observations together with previous lengthening pendulum experiments lead to the proposition stated in section 4. Namely that the linearly independent solutions or the amplitude and phase variables are related by the invariant relationship if the time-dependent parameter is finite. The proof of this proposition is given using a recent derivation of the orthogonal functions' invariant. The last section appraises some consequences of these results.

2. Experiments and numerical results

The lengthening pendulum is perhaps the simplest macroscopic mechanical apparatus that has been devised in order to experimentally study the time-dependent harmonic oscillator problem. Under adequate conditions it is possible to vary the length in an adiabatic or abrupt fashion. Recent experiments have given some insight on this system's behaviour, particularly in the regime where the problem is mathematically less tractable, that is, when the length varies in a time span that is in the order of the period [8]. For the present purposes, the result that we wish to highlight is that it is possible to execute an abrupt variation of the length while the amplitude and phase functions remain continuous.

The numerical evaluation of the TDHO equation (1) is complicated because the system's trajectory for constant parameter (harmonic oscillation) is interlaced with the behaviour produced by the time dependence of the parameter. Nonetheless, an amplitude and phase representation makes it easier to establish appropriate conditions since it is possible to separate the evolution coming from the two contributions.

Let the coordinate variable q be written in terms of amplitude A and phase s variables as

$$q(t) = A(t)\cos[s(t)]. \tag{2}$$

There is no general analytic solution to the TDHO equation (1) for an arbitrary parameter Ω^2. However, for a given trajectory $q(t)$ it is straightforward to obtain the parameter function

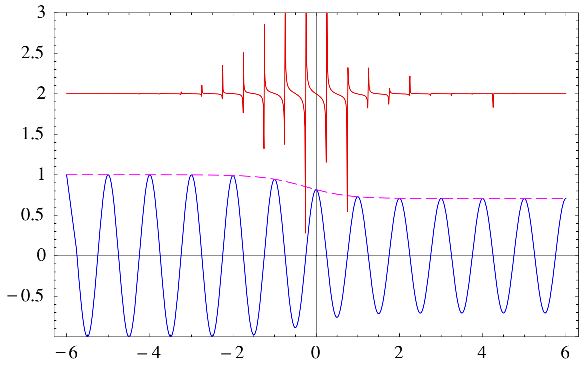

Figure 1. Coordinate q (blue lower solid line), amplitude A (dashed magenta) and time-dependent parameter Ω/Ω_0 (red upper curve) versus time for a constant frequency. *Invariant condition not imposed.*

that produces such a path since

$$\Omega^2(t) = -\frac{1}{q(t)}\frac{\mathrm{d}^2 q(t)}{\mathrm{d}t^2} = -\frac{1}{A\cos s}\frac{\mathrm{d}^2(A\cos s)}{\mathrm{d}t^2}. \quad (3)$$

Recall that so far the amplitude and phase functions may be arbitrarily chosen.

2.1. Attenuated amplitude

Example 1. Consider a system that acts as an attenuator for the disturbance, namely that the amplitude is decreased while the frequency is kept constant. Take, for example, an amplitude of the form

$$A = A_0\sqrt{2}[3 + \tanh(\alpha t)]^{-\frac{1}{2}}, \quad (4)$$

where α is a measure of how fast the amplitude changes. This function varies smoothly from A_0 at $t = -\infty$ to $A_0/\sqrt{2}$ at $t = \infty$, that is the intensity is halved. Let the frequency ω be constant so that the phase is $s = \omega t$. Using these values for the amplitude and phase, the time-varying parameter $\Omega^2(t)$ obtained from (3) is then

$$\frac{-3\alpha^2\cosh^{-4}\alpha t - 4\alpha(\tanh\alpha t + 3)(\omega\tan\omega t + \alpha\tanh\alpha t)\cosh^{-2}\alpha t + 4\omega^2(\tanh\alpha t + 3)^2}{4(\tanh\alpha t + 3)^2}.$$

This function together with the trajectory (coordinate) and amplitude functions is plotted in figure 1. The time-varying parameter exhibits divergent results when the wavefunction passes through the equilibrium position. These singularities are present even if the amplitude varies slowly over many periods. So that although the amplitude and phase are smooth continuous functions, the time-dependent parameter exhibits singularities where its value is infinite. The graph depicts these kinks with finite value due to numerical rounding, but if the region is magnified, ever-increasing values of the parameter are obtained.

2.2. Frequency modulation

Example 2. Allow for a system whose amplitude remains constant while its frequency is varied. Let the phase function be

$$s(t) = \frac{\omega\{3\alpha t + \log[\cosh(\alpha t)]\}}{2\alpha},$$

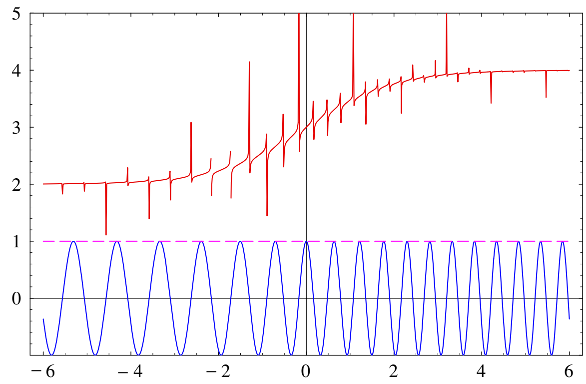

Figure 2. Coordinate q (blue lower solid line), amplitude A (dashed magenta) and time-dependent parameter Ω/Ω_0 (red upper curve) versus time for a constant amplitude. *Invariant condition not imposed*.

so that the frequency is doubled

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\omega(\tanh(\alpha t) + 3)}{2}.$$

The time-varying parameter from (3) is then

$$\Omega^2(t) = \frac{1}{4}\omega\left(2\alpha\tan\left(\frac{\omega[3\alpha t + \log(\cosh\alpha t)]}{2\alpha}\right)\cosh^{-2}\alpha t + \omega(\tanh\alpha t + 3)^2\right).$$

This function is plotted in figure 2. Again, $\Omega^2(t)$ has divergent singularities although the amplitude and phase functions are smooth.

2.3. Polynomial

Example 3. As a last example, we present the most simple functions that produce a time-dependent parameter. Allow for a system whose amplitude remains constant while its frequency increases linearly with time. Let the phase function be quadratic

$$s(t) = \omega t^2,$$

so that the frequency is linear in time

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 2\omega t.$$

The time-varying parameter from (3) is then

$$\Omega^2(t) = 2\omega(2\omega t^2 + \tan(t^2\omega)).$$

This function is plotted in figure 3. Again, Ω^2 reveals divergent singularities although the amplitude and frequency are smooth well-behaved functions. Numerous trials with different amplitude and phase functions not fulfilling the invariant relationship were estimated. The outcome always yielded divergent results in the time-dependent parameter.

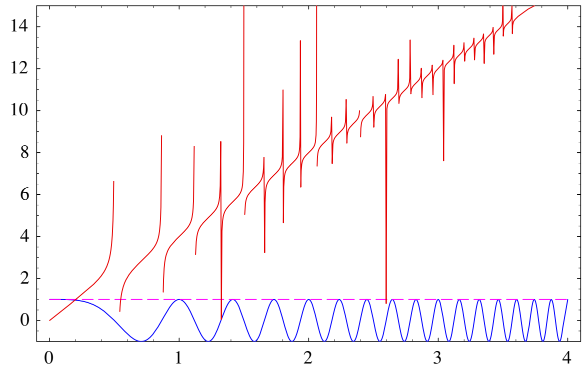

Figure 3. Coordinate q (blue lower solid line), amplitude A (dashed magenta) and time-dependent parameter Ω/Ω_0 (red upper curve) versus time for A, s polynomial functions. *Invariant condition not imposed.*

3. Exact invariant

Exact invariants for the time-dependent harmonic oscillator (TDHO) equation have been derived in a variety of ways including Kruskal theory, Noether's theorem, Ermakov's method and group transformation method [10]. A straightforward ansatz recently reported is the orthogonal functions' procedure [11]. This derivation is particularly well suited in order to prove the propositions stated in the next section. The invariant thus derived is given by

$$Q = q_1\dot{q}_2 - q_2\dot{q}_1 = A^2\frac{\mathrm{d}s}{\mathrm{d}t}, \tag{5}$$

where q_1, q_2 are linearly independent solutions to the TDHO equation. This invariant imposes a relationship between the amplitude and phase functions so that these variables are no longer independent. The equation for the coordinate variable is then separable into two uncoupled differential equations for the amplitude and the phase. The amplitude equation is

$$\ddot{A} + \Omega^2 A - \frac{Q^2}{A^3} = 0. \tag{6}$$

The coordinate (1) and amplitude (6) equations form an Ermakov pair. It is clear that neither the amplitude given by (4) together with a linear phase nor the constant amplitude and quadratic phase fulfil the invariant relationship (5). However, if the phase is obtained from the invariant relationship for the proposed amplitude (4), then

$$s = \int \frac{Q}{A^2}\,\mathrm{d}t = \frac{\omega\{3\alpha t + \log[\cosh(\alpha t)]\}}{2\alpha}. \tag{7}$$

The time-dependent parameter evaluated from (3) with the amplitude (4) and phase (7) fulfilling the invariant equation is

$$\frac{\omega^2\left[24 - \frac{6\alpha^2}{\omega^2}\sinh 2\alpha t - \frac{\alpha^2}{\omega^2} - \left(\frac{2\alpha^2}{\omega^2} - 40\right)\cosh 2\alpha t + 17\cosh 4\alpha t + 24\sinh 2\alpha t + 15\sinh 4\alpha t\right]}{4\cosh^4(t\alpha)(\tanh(t\alpha) + 3)^2}.$$

This function, plotted in figure 4, is now smooth and does not exhibit singularities. Some other simple solutions for unbounded parameters have been reported before without an explicit procedure for their deduction [9]. The above experience was repeated with different amplitude and phase class C^2 functions obtaining always analogous results. Namely, that the time-dependent parameter does not show singularities if the invariant relationship $Q = A^2\dot{s}$ is fulfilled.

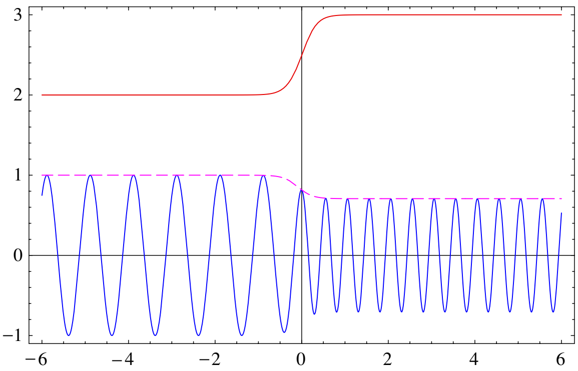

Figure 4. Coordinate q (blue lower solid line), amplitude A (dashed magenta) and time-dependent parameter Ω/Ω_0 (red upper curve) versus time. *Invariant condition fulfilled.*

4. Existence of invariant

These observations may be formulated in the following:

Proposition 1. *The time-dependent harmonic oscillator equation* $\frac{d^2q}{dt^2} + \Omega^2 q = 0$ *possesses an invariant* $Q = q_1\dot{q}_2 - q_2\dot{q}_1$, *where* q_1, q_2 *are the linearly independent solutions, provided that the time-dependent parameter* Ω^2 *is continuous and piecewise smooth over the entire time variable domain.*

Proof. Recall that the orthogonal functions' procedure evaluates the difference between two equations each of them constructed from the product of the TDHO equation with a given solution times the linearly independent solution [11]. The derivation leading to the invariant Q requires that

$$\frac{d}{dt}(q_1\dot{q}_2 - q_2\dot{q}_1) + \Omega^2(q_1q_2 - q_2q_1) = 0, \tag{8}$$

where q_1, q_2 are the linearly independent solutions to the TDHO equation (1). The invariant $Q = q_1\dot{q}_2 - q_2\dot{q}_1$ thus exists provided that the term $\Omega^2(q_1q_2 - q_2q_1)$ is zero for all possible values of t. The factor in brackets is zero for finite q_1, q_2. The term is then zero provided that the Ω^2 factor is finite. □

An equivalent assertion in terms of amplitude and phase variables is as follows:

Proposition 2. *If the time-dependent parameter* Ω^2 *is continuous and piecewise smooth in its time variable domain, then the time-dependent harmonic oscillator equation* $\frac{d^2q}{dt^2} + \Omega^2 q = 0$ *has an invariant* $Q = A^2\frac{ds}{dt}$, *where* A *and* s *correspond to the amplitude and phase representation of the coordinate variable* q.

Proof. The proof follows from the previous derivation allowing for an amplitude and phase representation

$$q_1 = A\cos(s), \qquad q_2 = A\sin(s)$$

that ensures linearly independent solutions. Recall that linearly independent solutions require that the Wronskian should be different from zero. Since the orthogonal functions' invariant in this particular case is equal to the Wronskian, then $A^2\dot{s} \neq 0$. The amplitude and frequency functions should then be different from zero,

$$A, \dot{s} \neq 0,$$

in order to obtain a non-vanishing invariant. □

4.1. Continuity

The coordinate and amplitude functions are at least class C^2 continuous since they are solutions to second-order differential equations (1) and (6). The phase function s has to be class C^3 as may be seen from the invariant relationship or from the third-order differential equation that it fulfils. However, it should be noted that the time-dependent parameter Ω^2 function may be discontinuous even if A and s are continuous. One such case is given by a step function that nonetheless gives continuous amplitude and phase analytical functions [7].

4.2. Complex representation

The real amplitude and phase representation of the coordinate variable (2) is often translated into a complex representation $q = A \exp(\mathrm{i}s)$ as an economical formalism for linear differential equations. However, this representation leads to two real-valued differential equations that necessarily impose the invariant relationship (5) [11]. Therefore, according to the previous results, the complex notation is not merely an alternative representation but is restricting the time-dependent parameter to continuous, piecewise smooth functions.

5. Conclusions

It has been proved that the time-dependent harmonic oscillator equation possesses an exact invariant provided that the time-dependent parameter Ω^2 is continuous and piecewise smooth. The exact invariant is valid even if Ω^2 varies abruptly. It has also been shown that the real amplitude and phase representation $q = A \cos s$ does not imply the invariant relationship $A^2\dot{s} =$ constant. If this equality is not fulfilled, the parameter Ω^2 exhibits infinite discontinuities. In contrast, if the invariant condition is fulfilled, the parameter is smooth and finite. The complex representation $q = A \exp(\mathrm{i}s)$ implicitly requires that the invariant relationship is enforced. The coordinate, amplitude and phase functions are at least class C^2 continuous since the invariant allows the decoupling of these variables' differential equations. Since any physically attainable time-dependent parameter must be finite and continuous, any real TDHO system must fulfil the invariant relationship.

Acknowledgment

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